

LOG CANONICAL THRESHOLDS ON SMOOTH VARIETIES: THE ASCENDING CHAIN CONDITION

LAWRENCE EIN AND MIRCEA MUSTĂŢĂ

ABSTRACT. Building on results of Kollár, we prove Shokurov's ACC Conjecture for log canonical thresholds on smooth varieties, and more generally, on varieties with quotient singularities.

1. INTRODUCTION

Let X be a smooth variety over an algebraically closed field k , of characteristic zero. If $f \in \mathcal{O}(X)$ is a nonzero regular function on X , and $p \in X$ is a point such that $f(p) = 0$, then the log canonical threshold $\text{lct}_p(f)$ is an invariant of the singularity at p of the hypersurface defined by f . This invariant plays a fundamental role in birational geometry. For an overview of the many contexts in which this invariant appears, and for some applications, see for example [EM]. The following is our main result.

Theorem 1.1. *For every $n \geq 1$, the set*

$$\mathcal{T}_n := \{\text{lct}_p(f) \mid f \in \mathcal{O}(X), f(p) = 0, X \text{ smooth}, \dim(X) = n\}$$

satisfies the Ascending Chain Condition, that is, it contains no infinite strictly increasing sequences.

The above statement has been conjectured by Shokurov in [Sho]. In fact, Shokurov's conjecture is in a more general setting, allowing the ambient variety to have mild singularities. The interest in this conjecture comes from the fact that its general singular version is related to the Termination of Flips Conjecture (see [Bir] for a statement in this direction). While the above theorem does not have such strong consequences, we believe that it offers strong evidence for the general case of the conjecture. Moreover, it suggests that this general case might not be out of reach.

As we will see in Proposition 3.4 below, every log canonical threshold on a variety with quotient singularities can be also written as a log canonical threshold on a smooth variety of the same dimension. Therefore the above theorem also implies Shokurov's Conjecture for log canonical thresholds on varieties with quotient singularities.

The first unconditional results on limit points of log canonical thresholds in arbitrary dimension have been proved in [dFM], using ultrafilter constructions. The key point was to

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show that given a sequence of polynomials $f_m \in k[x_1, \dots, x_n]$ such that $\lim_{m \rightarrow \infty} \text{lct}_0(f_m) = c$, one can construct a formal power series $f \in K[[x_1, \dots, x_n]]$ (for a suitable field extension K of k) such that $\text{lct}_0(f) = c$ (one can define log canonical thresholds also for formal power series, and it seems that for questions involving limit points of log canonical thresholds, this is the right framework).

The results in *loc. cit.* have been reproved by Kollár in [Koll1], replacing the ultrafilter construction by a purely algebro-geometric one. His approach turned out to be better suited for ruling out increasing sequences of log canonical thresholds. In fact, using a deep recent result in the Minimal Model Program from [BCHM], he showed that if (with the notation in the previous paragraph) the formal power series f is such that its log canonical threshold is computed by a divisor with center at the origin, then $\text{lct}_0(f_m) \geq c$ for all m large enough. In this note we show that elementary arguments on log canonical thresholds allow us to reduce to this special case, and therefore prove Theorem 1.1 above.

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2. GETTING ISOLATED LOG CANONICAL CENTERS

Let k be an algebraically closed field of characteristic zero. Fix $n \geq 1$, and let $R = k[[x_1, \dots, x_n]]$. We put $X = \text{Spec}(R)$. Suppose that $f \in R$ is a nonzero element in the maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. Our goal in this section is to show that after possibly replacing f by a suitable f^r , we can find a polynomial $g \in k[x_1, \dots, x_n]$ such that $\text{lct}_0(f) = \text{lct}_0(fg)$ and such that the log canonical threshold of fg can be computed by a divisor E over X with center at the closed point. Our main reference for log canonical thresholds in the formal power series setting is [dFM] (see also [Koll2], §8 for the basic facts in the more familiar setting of varieties over k).

Let us fix a log resolution $\pi: Y \rightarrow X$ for the pair $(R, f \cdot \mathfrak{m})$. Note that by the main result in [Tem] such resolutions exist, and this is what allowed in [dFM] the extension of the usual results on log canonical thresholds to our setting. We write $f \cdot \mathcal{O}_Y = \mathcal{O}(-\sum_i a_i E_i)$, $\mathfrak{m} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-\sum_i b_i E_i)$, and $K_{Y/X} = \mathcal{O}_Y(-\sum_i k_i E_i)$. Note that the center of E_i on X is the closed point (that we denote by 0) if and only if $b_i > 0$.

The log canonical threshold of f is given by

$$\text{lct}_0(f) = \min_i \frac{k_i + 1}{a_i}.$$

We say that a divisor E_j computes the log canonical threshold $\text{lct}_0(f)$ if $\text{lct}_0(f) = \frac{k_j + 1}{a_j}$. Let I denote the set of those j for which E_j has center equal to the closed point. If there is $j \in I$ computing $\text{lct}_0(f)$, then there is nothing we need to do. We now assume that this is not the case. The following is our key observation.

Proposition 2.1. *With the above notation, suppose that $c := \text{lct}_0(f) < \frac{k_i+1}{a_i}$ for every $i \in I$. If*

$$(1) \quad q := \min \left\{ \frac{1}{b_i} \left(\frac{k_i+1}{c} - a_i \right) \mid i \in I \right\},$$

then $\text{lct}_0(f \cdot \mathfrak{m}^q) = \text{lct}_0(f)$, and there is a divisor E over X computing $\text{lct}_0(f \cdot \mathfrak{m}^q)$, with center equal to the closed point.

Proof. Note that our assumption implies $q > 0$. By construction π is also a log resolution of $(X, f \cdot \mathfrak{m}^q)$. Using the above notation for $f \cdot \mathcal{O}_Y$, $\mathfrak{m} \cdot \mathcal{O}_Y$, and $K_{Y/X}$, we see that

$$\text{lct}_0(f \cdot \mathfrak{m}^q) = \min_i \left\{ \frac{k_i+1}{a_i + qb_i} \right\}.$$

Note that if $i \notin I$, then $b_i = 0$ and $\frac{k_i+1}{a_i+qb_i} = \frac{k_i+1}{a_i} \geq c$.

On the other hand, if $i \in I$ then by the definition of q we have

$$(2) \quad \frac{k_i+1}{a_i + qb_i} \geq c.$$

This shows that $\text{lct}_0(f \cdot \mathfrak{m}^q) \geq c$. Moreover, if $i \in I$ is such that the minimum in (1) is achieved, then we have equality in (2). This shows that $\text{lct}_0(f \cdot \mathfrak{m}^q) = \text{lct}_0(f)$, and there is a divisor E_i with center equal to the closed point, that computes $\text{lct}_0(f \cdot \mathfrak{m}^q)$. \square

Our next goal is to replace the ideal \mathfrak{m}^q by the suitable power of a product of very general linear forms. Suppose that \mathfrak{a} is an ideal in $k[x_1, \dots, x_n]$ generated by h_1, \dots, h_d , and such that $\text{lct}_0(\mathfrak{a}) = c$. It is well-known that if $N \geq c$ and g_1, \dots, g_N are general linear combinations of h_1, \dots, h_d with coefficients in k , then $\text{lct}_0(g) = c/N$, where $g = g_1 \cdot \dots \cdot g_N$ (see, for example, Prop. 9.2.26 in [Laz] for a related statement). The proof of this result, however, relies on Bertini's Theorem, so it does not simply carry over to our setting. When dealing with formal power series, we will argue by taking truncations. Let us start with a small variation on the above-mentioned result.

Lemma 2.2. *Let \mathfrak{a} and \mathfrak{b} be nonzero ideals in the polynomial ring $k[x_1, \dots, x_n]$, and $\alpha, \beta > 0$ such that $\text{lct}_0(\mathfrak{a}^\alpha \cdot \mathfrak{b}^\beta) = c_0$. Let N be a positive integer such that $N/\beta \geq c_0$. If h_1, \dots, h_d generate \mathfrak{b} , and if g_1, \dots, g_N are general linear combinations of the h_i with coefficients in k , then $\text{lct}_0(\mathfrak{a}^\alpha \cdot g^{\beta/N}) = c_0$, where $g = g_1 \cdot \dots \cdot g_N$.*

Proof. The argument is the same one as in *loc. cit.*, but we recall it for completeness. Let $\mu: W \rightarrow \mathbf{A}^n$ be a log resolution of $\mathfrak{a} \cdot \mathfrak{b}$. Let us write $\mathfrak{a} \cdot \mathcal{O}_W = \mathcal{O}(-\sum_i u_i E_i)$, $\mathfrak{b} \cdot \mathcal{O}_W = \mathcal{O}(-\sum_i w_i E_i)$, and $K_{W/\mathbf{A}^n} = \sum_i k_i E_i$. Therefore

$$c_0 = \min_{i \in J} \left\{ \frac{k_i+1}{\alpha u_i + \beta w_i} \right\},$$

where J is the set of those i such that $0 \in \mu(E_i)$.

Since each g_j is a general combination of the generators of \mathfrak{b} , it follows by Bertini's Theorem that $g_j \cdot \mathcal{O}_W = \mathcal{O}(-F_j - \sum_i w_i E_i)$, where F_1, \dots, F_N are divisors with no common components amongst them or with the E_i , such that $\sum_i E_i + \sum_j F_j$ has simple normal

crossings. In particular, μ is a log resolution of $\mathbf{a} \cdot g$. Since $g \cdot \mathcal{O}_W = \mathcal{O}(-\sum_i N w_i E_i - \sum_j F_j)$, and

$$\min \left\{ \min_{i \in J} \frac{k_i + 1}{\alpha u_i + \beta w_i}, \frac{1}{\beta/N} \right\} = c_0$$

(we use the hypothesis that $N/\beta \geq c_0$), it follows that $\text{lct}_0(\mathbf{a}^\alpha \cdot g^{\beta/N}) = c_0$. \square

In the following proposition we assume that the ground field k is uncountable. In this case, we follow the standard terminology by saying that a very general point on an irreducible algebraic variety over k is a point that lies outside a countable union of proper subvarieties.

Proposition 2.3. *With the notation in Proposition 2.1, let $N > qc$ be an integer. If g_1, \dots, g_N are very general linear forms with coefficients in k , and $g = g_1 \cdots g_N$, then $\text{lct}_0(f \cdot g^{q/N}) = c$, and there is a divisor E over X computing $\text{lct}_0(f \cdot g^{q/N})$, with center equal to the closed point.*

Proof. For every ℓ , let $f_{\leq \ell}$ denote the truncation of f of degree $\leq \ell$. Let us put $c_\ell = \text{lct}_0(f_{\leq \ell} \cdot \mathbf{m}^q)$. It follows from Proposition 2.1 and Prop. 2.5 in [dFM] that $\lim_{\ell \rightarrow \infty} c_\ell = c$. In particular, if $\ell \gg 0$, then $N > qc_\ell$. For such ℓ we may apply the lemma to deduce that $\text{lct}_0(f_{\leq \ell} \cdot g^{q/N}) = c_\ell$ (note the since we are allowed to take the g_i as very general linear combinations of x_1, \dots, x_n , we can simultaneously apply the lemma for all ℓ as above). Another application of Prop. 2.5 in [dFM] now gives $\text{lct}_0(f \cdot g^{q/N}) = c$.

Suppose that E is the divisor given by Proposition 2.1, so it computes $\text{lct}_0(f \cdot \mathbf{m}^q)$, and its center is equal to the closed point. We denote by ord_E the valuation of the fraction field of R corresponding to E . Since $g \in \mathbf{m}^N$, we deduce $\text{ord}_E(f \cdot g^{q/N}) \geq \text{ord}_E(f \cdot \mathbf{m}^q)$. Therefore we have

$$(3) \quad c = \frac{\text{ord}_E(K_{Y/X}) + 1}{\text{ord}_E(f \cdot \mathbf{m}^q)} \geq \frac{\text{ord}_E(K_{Y/X}) + 1}{\text{ord}_E(f \cdot g^{q/N})} \geq c,$$

where the second equality follows from the fact that $\text{lct}_0(f \cdot g^{q/N}) = c$. Therefore both inequalities in (3) are equalities, and we see that E computes $\text{lct}_0(f \cdot g^{q/N})$. \square

3. THE PROOF OF THE ACC CONJECTURE

Before giving the proof of Theorem 1.1, let us describe the key construction from [Koll]. For every nonnegative integer m , let $\text{Pol}_{\leq m}$ denote the affine space $\mathbf{A}_k^{N_m}$ (with $N_m = \binom{n+m}{m}$), such that for every field extension K of k , the K -rational points of $\text{Pol}_{\leq m}$ parametrize the polynomials in $K[x_1, \dots, x_n]$ of degree $\leq m$. We have obvious maps $\pi_m: \text{Pol}_{\leq m} \rightarrow \text{Pol}_{\leq (m-1)}$ that correspond to truncation of polynomials. Given a field extension K of k , a formal power series $f \in K[[x_1, \dots, x_n]]$ corresponds to a sequence of morphisms $\text{Spec } K \rightarrow \text{Pol}_{\leq m}$ over $\text{Spec } k$, compatible via the truncation maps. We denote by $t_m(f)$ the corresponding element of $\text{Pol}_{\leq m}(K)$.

Suppose now that $(f_q)_q$ is a sequence of formal power series in $k[[x_1, \dots, x_n]]$, all being nonzero, and of positive order. We consider sequences of irreducible closed subsets $Z_m \subseteq \text{Pol}_{\leq m}$ with the following properties:

- i) For every m , there are infinitely many q such that $t_m(f_q) \in Z_m$.
- ii) Z_m is the Zariski closure of those $t_m(f) \in Z_m$.
- iii) Each truncation morphism π_m induces a dominant morphism $Z_m \rightarrow Z_{m-1}$.

Such sequences can be constructed by induction. We start the induction by taking $Z_0 = \text{Pol}_{\leq 0} = \text{Spec}(k)$. If Z_m is constructed, then we take Z_{m+1} to be a minimal irreducible closed subset of $\pi_{m+1}^{-1}(Z_m)$ with the property that it contains $t_{m+1}(f_q)$ for infinitely many q . Properties i)-iii) are clear: note that the minimality assumption in the definition of Z_{m+1} implies that the induced morphism $Z_{m+1} \rightarrow Z_m$ is dominant.

Suppose that $(Z_m)_m$ is a sequence satisfying i)-iii) above. Let η_m denote the generic point of Z_m , so the truncation maps induce embeddings $k(\eta_m) \hookrightarrow k(\eta_{m+1})$. If K is an algebraically closed field containing $\bigcup_m k(\eta_m)$, we get corresponding maps $\text{Spec}(K) \rightarrow \text{Pol}_{\leq m}$ that are compatible via the truncation morphisms. This corresponds to a formal power series $f \in K[[x_1, \dots, x_n]]$ such that $t_m(f)$ gives η_m . Of course, this construction is not unique. However, whenever f is obtained from the sequence $(f_q)_q$ via such a sequence $(Z_m)_m$, we say that f is a *generic limit* of $(f_q)_q$. A trivial example is when $f_m = h$ for every m , in which case each Z_m is a point, and for every field extension K of k , a generic limit of this sequence is given by the image of f in $K[[x_1, \dots, x_n]]$.

A key property of generic limits that follows from construction is that for every m , there are infinitely many q such that $\text{lct}_0(t_m(f)) = \text{lct}_0(t_m(f_q))$. It is easy to deduce from this that if $\lim_{q \rightarrow \infty} \text{lct}_0(f_q) = c$, then $\text{lct}_0(f) = c$ (see Thm. 29 in [Kol1] for details).

We start with an easy lemma describing the behavior of generic limits under multiplication.

Lemma 3.1. *Let $(f_q)_q$ and $(h_q)_q$ be two sequences of formal power series in $k[[x_1, \dots, x_n]]$. If $f \in K'[[x_1, \dots, x_n]]$, and $h \in K''[[x_1, \dots, x_n]]$ are generic limits of $(f_q)_q$, and respectively $(h_q)_q$, then there is a field extension K of both K' and K'' , such that $fh \in K[[x_1, \dots, x_n]]$ is a generic limit of the sequence $(f_q h_q)_q$.*

Proof. Suppose that $Z'_m, Z''_m \subseteq \text{Pol}_{\leq m}$ are sequences of subsets satisfying i)-iii) above, with respect to $(f_q)_q$, and respectively $(h_q)_q$. If η'_m and η''_m are the generic points of respectively Z'_m and Z''_m , then K' contains all $k(\eta'_m)$, and similarly, K'' contains all $k(\eta''_m)$. For every m we have a morphism $\varphi_m: \text{Pol}_{\leq m} \times \text{Pol}_{\leq m} \rightarrow \text{Pol}_{\leq m}$ that corresponds to multiplication, followed by truncation up to degree m . It is clear that we have $\pi_m \circ \varphi_m = \varphi_{m-1} \circ (\pi_m, \pi_m)$ for every positive m . If we take $Z_m := \overline{\varphi_m(Z'_m \times Z''_m)}$, then $(Z_m)_m$ is a sequence of irreducible closed sets that satisfies i)-iii) with respect to the sequence $(f_q h_q)_q$. Moreover, if η_m is the generic point of Z_m , then we have embeddings compatible with the maps induced by truncation

$$k(\eta_m) \hookrightarrow Q(k(\eta'_m) \otimes_k k(\eta''_m)),$$

where for a domain S we denote by $Q(S)$ its fraction field. Since k is algebraically closed, $K' \otimes_k K''$ is a domain. If K is an algebraically closed extension of $Q(K' \otimes_k K'')$, then K contains $\bigcup_m k(\eta_m)$, and it follows from the construction that fh is the generic limit corresponding to the sequence $(Z_m)_m$, and to the embedding $\bigcup_m k(\eta_m) \hookrightarrow K$. \square

The following is the key result of Kollár that we will use. Its proof uses the difficult finite generation result of [BCHM].

Proposition 3.2. (Prop. 40, [Koll1]) *Let $K \supseteq k$ be a field extension, and suppose that $F \in K[[x_1, \dots, x_n]]$ is a formal power series. Suppose that there is a divisor computing $\text{lct}_0(F)$ with center at the closed point. If $Z_m \subseteq \text{Pol}_{\leq m}$ is the k -Zariski closure of $t_m(F)$, then there is a positive integer m , and an open subset $U_m \subseteq Z_m$ such that for every power series $G \in K[[x_1, \dots, x_n]]$ with $t_m(G) \in U_m$, we have $\text{lct}_0(F) = \text{lct}_0(G)$.*

Note that if F is a generic limit of a sequence $(f_q)_q$ of power series in $k[[x_1, \dots, x_n]]$, then the sets Z_m in the above statement are the same as the sets that come up in the definition of F as a generic limit. We can now prove our main result.

Proof of Theorem 1.1. We may assume that k is uncountable, and it is enough to show that the set

$$\mathcal{T}'_n = \{\text{lct}_0(f) \mid f \in k[[x_1, \dots, x_n]], f \neq 0, \text{ord}(f) \geq 1\}$$

contains no strictly increasing infinite sequences. This follows since we can extend scalars to an uncountable algebraically closed field, and we can complete at the given point $p \in X$ (in fact, it is shown in [dFM] that $\mathcal{T}_n = \mathcal{T}'_n$, and that this set is independent of the algebraically closed field k ; however, we do not need this fact).

Let us suppose that $(f_q)_q$ is a sequence of formal power series in $k[[x_1, \dots, x_n]]$, nonzero and of positive order, such that if we put $c_q = \text{lct}_0(f_q)$, then the corresponding sequence $(c_q)_q$ is strictly increasing. We put $c := \lim_{q \rightarrow \infty} c_q$. Let $f \in K[[x_1, \dots, x_n]]$ be a generic limit of the sequence $(f_q)_q$, with K an algebraically closed extension of k . As mentioned above, we have $\text{lct}_0(f) = c$.

If $\text{lct}_0(f)$ is computed by some divisor E with center equal to the closed point, then we are done. Indeed, we apply Proposition 3.2 above to $F = f$ to get m and an open subset $U_m \subseteq Z_m$ with the property that whenever $G \in K[[x_1, \dots, x_n]]$ satisfies $t_m(G) \in U_m$, we have $\text{lct}_0(f) = \text{lct}_0(G)$. On the other hand, by assumption Z_m is the Zariski closure of those $t_m(f_q) \in Z_m$. Therefore we can find q with $t_m(f_q) \in U_m$, so that $c_q = c$, a contradiction.

Suppose now that $\text{lct}_0(f)$ is not computed by any divisor with center equal to the closed point. In this case Propositions 2.1 and 2.3 imply that we can find a positive rational number q , a positive integer N and $g \in k[[x_1, \dots, x_n]]$ such that $\text{lct}_0(f \cdot g^{q/N}) = c$, and there is a divisor E computing $\text{lct}_0(f \cdot g^{q/N})$, and having center equal to the closed point. Let us write $q/N = s/r$, for positive integers r and s . It is clear that $\text{lct}_0(f^r g^s) = c/r$, and the same divisor E computes this log canonical threshold.

On the other hand, by Lemma 3.1 the power series $f^r g^s$ is a generic limit of the sequence $(f_q^r g^s)_q$. Since $\text{lct}_0(f^r g^s)$ is computed by a divisor with center equal to the closed point, using Proposition 3.2 for $F = f^r g^s$, we deduce that for some q we have $\text{lct}_0(f_q^r g^s) = c/r$. Note now that $\text{lct}_0(f_q^r g^s) \leq \text{lct}_0(f_q^r) = c_q/r$, so we have a contradiction. This completes the proof of the theorem. \square

Remark 3.3. While we stated Theorem 1.1 only for log canonical thresholds of principal ideals, it is straightforward to extend the statement to arbitrary ideals. More precisely,

the theorem implies that for every $n \geq 1$, the set of all log canonical thresholds $\text{lct}_p(X, \mathfrak{a})$ satisfies the Ascending Chain Condition, when X varies over the smooth n -dimensional varieties, and \mathfrak{a} over the nonzero ideals on X containing p in their support. Indeed, we may assume that X is affine, in which case if f_i is a general linear combination of the generators of \mathfrak{a} , for $1 \leq i \leq n$, and if $f = f_1 \cdots f_n$, then $\frac{1}{n} \cdot \text{lct}_p(X, \mathfrak{a}) = \text{lct}_p(X, f)$ (note that $\text{lct}_p(X, \mathfrak{a}) \leq n$, and apply for example Lemma 2.2). Therefore each such $\text{lct}_p(X, \mathfrak{a})$ lies in $n \cdot \mathcal{T}_n$, and this set satisfies the Ascending Chain Condition by Theorem 1.1.

The following proposition allows us to reduce log canonical thresholds on varieties with quotient singularities to log canonical thresholds on smooth varieties. We say that a variety X has quotient singularities at $p \in X$ if there is a smooth variety U , a finite group G acting on U , and a point $q \in V = U/G$ such that the two completions $\widehat{\mathcal{O}_{X,p}}$ and $\widehat{\mathcal{O}_{V,q}}$ are isomorphic as k -algebras. One can assume that U is an affine space and that the action of G is linear. Furthermore, one can assume that G acts with no fixed points in codimension one (otherwise, we may replace G by G/H and U by U/H , where H is generated by all pseudoreflections in G , and by Chevalley's theorem [Che], the quotient U/H is again an affine space). Using Artin's approximation results (see Corollary 2.6 in [Art]), it follows that there is an étale neighborhood of p that is also an étale neighborhood of q . In other words, there is a variety W , a point $r \in W$, and étale maps $\varphi: W \rightarrow X$ and $\psi: W \rightarrow V$, such that $p = \varphi(r)$ and $q = \psi(r)$. After replacing φ by the composition

$$W \times_V U \rightarrow W \xrightarrow{\varphi} X,$$

we may assume that in fact we have an étale map $U/G \rightarrow X$ containing p in its image, with U smooth, and such that G acts on U without fixed points in codimension one. This reinterpretation of the definition of quotient singularities seems to be well-known to experts, but we could not find an explicit reference in the literature.

We say that X has quotient singularities if it has quotient singularities at every point.

Proposition 3.4. *Let X be a variety with quotient singularities, and let \mathfrak{a} be a proper nonzero ideal on X . For every p in the zero-locus $V(\mathfrak{a})$ of \mathfrak{a} , there is a smooth variety U , a nonzero ideal \mathfrak{b} on U , and a point q in $V(\mathfrak{b})$ such that $\text{lct}_p(X, \mathfrak{a}) = \text{lct}_q(U, \mathfrak{b})$.*

Proof. Let us choose an étale map $\varphi: U/G \rightarrow X$ with $p \in \text{Im}(\varphi)$, where U is a smooth variety, and G is a finite group acting on U without fixed points in codimension one. Let $\tilde{\varphi}: U \rightarrow X$ denote the composition of φ with the quotient map. Since G acts without fixed points in codimension one, $\tilde{\varphi}$ is étale in codimension one, hence $K_U = \tilde{\varphi}^*(K_X)$. It follows from Proposition 5.20 in [KM] that if $\mathfrak{b} = \mathfrak{a} \cdot \mathcal{O}_U$, then the pair (X, \mathfrak{a}^q) is log canonical if and only if the pair (U, \mathfrak{b}^q) is log canonical (actually the result in *loc. cit.* only covers the case when \mathfrak{a} is locally principal, but one can easily reduce to this case, by taking a suitable product of general linear combinations of the local generators of \mathfrak{a}). We conclude that there is a point $q \in V(\mathfrak{b})$ such that $\text{lct}_p(X, \mathfrak{a}) = \text{lct}_q(U, \mathfrak{b})$. \square

Remark 3.5. At least over the complex numbers, one usually says that X has quotient singularities at p if the germ of analytic space (X, x) is isomorphic to M/G , where M is

a complex manifold, and G is a finite group acting on M . It is not hard to check that in this context this definition is equivalent with the one we gave above.

Combining Proposition 3.4 and Theorem 1.1 (see also Remark 3.3), we deduce Shokurov's ACC Conjecture on varieties with quotient singularities.

Corollary 3.6. *For every $n \geq 1$, the set*

$$\{\mathrm{lct}_p(X, \mathfrak{a}) \mid X \text{ has quotient singularities, } \dim(X) = n, (0) \neq \mathfrak{a}, p \in V(\mathfrak{a})\}$$

satisfies the Ascending Chain Condition.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO, 851 SOUTH MORGAN STREET (M/C 249), CHICAGO, IL 60607-7045, USA

E-mail address: ein@math.uic.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, 530 CHURCH STREET, ANN ARBOR, MI 48109, USA

E-mail address: mmustata@umich.edu